A Unified Approach for Approximating 2-Edge-Connected Spanning Subgraph and 2-Vertex-Connected Spanning Subgraph

Ali Çivril*

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Abstract

We provide algorithms for the minimum 2-edge-connected spanning subgraph problem and the minimum 2-vertex-connected spanning subgraph problem with approximation ratio both $\frac{4}{3}$. Using a common theme, the algorithms and their analyses are very similar.

1 Introduction

We consider two fundamental connectivity problems, namely the minimum 2-edge-connected spanning subgraph problem (2-ECSS) and the minimum 2-vertex-connected spanning subgraph problem (2-VCSS).

In 2-ECSS, given an undirected simple graph $G = (V, E)$, one finds a 2-edge-connected spanning subgraph (2-ECSS) of G with minimum number of edges. The problem remains NP-hard and MAX SNP-hard even for subcubic graphs [7]. The first result beating the approximation factor 2 came from Khuller and Vishkin [15], which is a $\frac{3}{2}$-approximation algorithm. Cheriyan, Sebö and Szigeti [2] improved the factor to $\frac{17}{12}$. Vempala and Vetta [19], and Jothi, Raghavachari and Varadarajan [14] claimed to have $\frac{4}{3}$ and $\frac{5}{4}$ approximations, respectively. Krysta and Kumar [16] went on to give a $(\frac{4}{3} - \epsilon)$-approximation for some small $\epsilon > 0$ assuming the result of Vempala and Vetta [19]. A relatively recent paper by Sebö and Vygen [18] provides a $\frac{4}{3}$-approximation algorithm by using ear decompositions, and mentions that the aforementioned claimed approximation ratio $\frac{5}{4}$ has not appeared with a complete proof in a fully refereed publication. The result of Vempala and Vetta [19], which was initially incomplete, recently re-appeared in [13]. Given this, it is not clear if the ratio $(\frac{4}{3} - \epsilon)$ by Krysta and Kumar [16] still holds. Very recently, Garg, Grandoni, and Ameli [9] improved upon the ratio $\frac{4}{3}$ by a constant $\frac{1}{130} > \epsilon > \frac{1}{140}$, which stands as the best current result for approximating the problem.

In 2-VCSS, given an undirected simple graph $G = (V, E)$, one finds a 2-vertex-connected (or simply 2-connected) spanning subgraph (2-VCSS) of G with minimum number of edges. This problem is NP-hard via a reduction from the Hamiltonian cycle problem. Furthermore, by the result of Czumaj and Lingas [8], it does not admit a PTAS unless P = NP. The first result improving the factor 2 for 2-VCSS came from Khuller and Vishkin [15], which is a $\frac{5}{3}$-approximation algorithm. Garg, Vempala and Singla [10] improved the approximation ratio to $\frac{3}{2}$. Cheriyan and Thurimella [3] also provides the same approximation ratio in a more general context, where they consider k-connectivity. Vempala and Vetta [19] claimed the ratio $\frac{4}{3}$, which is shown to be not

*Atlas University, Computer Engineering Department, Kagithane, 34408 Istanbul, Turkey, e-mail: ali.civril@atlas.edu.tr

valid by Heeger and Vygen [12]. Jothi, Raghavachari and Varadarajan [14] claimed the ratio $\frac{5}{4}$. However, this claim has been later withdrawn (see [11]). Gubbala and Raghavachari [11] claimed to have a $\frac{9}{7}$-approximation algorithm. The only complete (and exceedingly long) proof of this claim is in Gubbala's thesis [17], which has not appeared anywhere else. To the best of our knowledge, the algorithm by Heeger and Vygen [12] with factor $\frac{10}{7}$ stands as the first refereed improvement over the factor $\frac{3}{2}$ after a long hiatus. This paper also contains a somewhat more detailed discussion about the aforementioned claimed results, implying that theirs is the first improvement. Very recently, Bosch-Calvo, Grandoni, and Ameli provided an algorithm with approximation ratio $\frac{4}{3}$ [1], the current best factor.

Although the two problems we consider are closely related to each other, as noted above, the work on these problems have generally appeared in distinct publications. The reason is that the solution being 2-edge-connected or 2-connected enforces different approaches if the algorithmic ideas are tightly coupled with the structure of the solution. We give a unified strategy to solve these problems, where we use an algorithmic idea almost oblivious to whether the solution should be 2-edge-connected or 2-connected. The starting point is [6] in which we gave an algorithm for 2-ECSS. Unfortunately, its analysis therein is not clear, and contains several inconsistencies, even with the corrigendum published after the paper [5]. The purpose of this paper is twofold: We provide a new algorithm for 2-ECSS, which starts from the ideas employed in [6], but with a clear analysis, thereby obtaining the ratio $\frac{4}{3}$. Secondly, we show that the same algorithmic idea can be applied to 2-VCSS to attain the same approximation ratio $\frac{4}{3}$, matching the current best approximation ratio. Both the algorithms and their analyses are considerably simpler compared to previous approaches. In essence, we do not use anything but a recursive local search over an inclusion-wise minimal starting solution.

2 Preliminaries

We will use the lower bound derived from the dual of the natural LP relaxation for 2-ECSS. Here, $\delta(S)$ denotes the set of edges with one end in the cut S and the other not in S.

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} x_e & & \text{(EC)} \\ \text{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq 2, & \forall\, \emptyset \subset S \subset V, & \\ & 1 \geq x_e \geq 0, & \forall\, e \in E. & \end{aligned}$$

The following is the dual of (EC).

$$\begin{aligned} \text{maximize} \quad & \sum_{\emptyset \subset S \subset V} 2y_S - \sum_{e \in E} z_e & & \text{(EC-D)} \\ \text{subject to} \quad & \sum_{S: e \in \delta(S)} y_S \leq 1 + z_e, & \forall e \in E, & \\ & y_S \geq 0, & \forall\, \emptyset \subset S \subset V, & \\ & z_e \geq 0, & \forall\, e \in E. & \end{aligned}$$

We now give the LP relaxation for 2-VCSS from [4], which will also be of use in one of our arguments. Set $n := |V|$, $m := |E|$. A setpair is an ordered pair of sets $W = (W_t, W_h)$ such that

$W_t \subseteq V$, $W_h \subseteq V$ and $W_t \cap W_h = \emptyset$. We say that an edge $(u, v) \in E$ *covers* W if $u \in W_t$, $v \in W_h$ or $v \in W_t$, $u \in W_h$. Let $\delta(W)$ denote the set of all edges in E that cover W, and $\mathcal{S}$ denote the set of all setpairs (W_t, W_h) such that W_t and W_h are non-empty. The following is an LP relaxation for 2-VCSS.

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} x_e & & \text{(VC)} \\ \text{subject to} \quad & \sum_{e \in \delta(W)} x_e \geq 2, & W \in \mathcal{S}, |W_t \cup W_h| = n, & \\ & \sum_{e \in \delta(W)} x_e \geq 1, & W \in \mathcal{S}, |W_t \cup W_h| = n-1, & \\ & 1 \geq x_e \geq 0, & \forall\, e \in E. & \end{aligned}$$

The algorithm we will present is common for 2-ECSS and 2-VCSS except a few points. To this aim, we assume that the input graph G is 2-connected. Otherwise, the algorithm of the next section can be executed on blocks (maximal 2-connected subgraphs) of G separately. This is without loss of generality, since in that case the value of an optimal solution for 2-ECSS is the sum of those of blocks, and we can argue the approximation ratio only within a block. The algorithm refers to the feasibility of a given solution depending on the type of the problem, i.e., either a 2-ECSS or a 2-VCSS. Given a vertex $v \in V$ and a feasible solution F, the degree of v on F is denoted by $deg_F(v)$. The vertex v is called a *degree-d vertex* on F if $deg_F(v) = d$, and a *high-degree vertex* on F if $deg_F(v) \geq 3$. For a path $P = v_1 v_2 \dots v_{k-1} v_k$, v_1 and v_k are the *end vertices* of P, and all the other vertices are the *internal vertices* of P. A path whose internal vertices are all degree-2 vertices on F is called a *plain path* on F. A maximal plain path is called a *segment*. The length of a segment is the number of edges on the segment. If the length of a segment is ℓ, it is called an *ℓ-segment*. A 1-segment is also called a *trivial segment*. If a feasible solution remains feasible upon removal of a trivial segment, the trivial segment is called *redundant*. For $\ell > 1$, an ℓ-segment is called a *non-trivial segment*. A non-trivial ℓ-segment with $\ell \leq 3$ is called a *short segment*, otherwise a *long segment*. If the removal of a segment from F violates feasibility, it is called a *weak segment* on F, otherwise a *strong segment* on F.

3 The Algorithm for 2-ECSS and 2-VCSS

The algorithm starts by considering an inclusion-wise minimal 2-VCSS on G (Recall our assumption from the previous section). This can be computed by taking all the edges in E, and then deleting an element of this set one by one as long as the feasibility is not violated. Let F be such a solution. The algorithm recursively modifies the running solution F via *improvement processes* to eliminate specific sets of edges from $E \setminus F$. More formally, given a strong short segment S and an internal vertex u of S, let $N(u) \subseteq E \setminus F$ denote the set of edges incident to u, which are not in F. In each iteration of a loop, the algorithm checks for a selected S and u if including a set H of k edges from $N(u)$, which we call a *critical edge set*, and excluding a $k+1$ edges from F maintains feasibility, where $k \in \{1, 2\}$. It switches to this cheaper feasible solution if it does, and this is called an *improvement operation*, which improves the cost of the solution by 1. Note that critical edge sets can be examined in polynomial time, as their sizes are constant. For explicitness, we list all the 5 types of improvement operations and the corresponding critical edge sets in Figure 1-Figure 5. The first two figures are specific to a 2-VCSS.

If there are no $k+1$ edges in F whose removal maintains feasibility, fixing the set H of edges in $N(u)$ to include, the algorithm checks if $F \cup H$ contains new strong short segments that do not

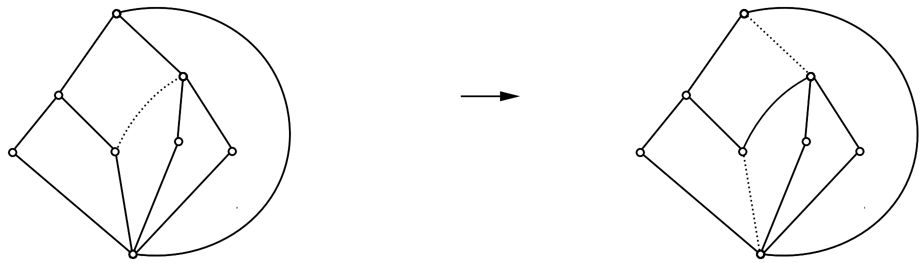

Figure 1: An example of an improvement operation of Type I

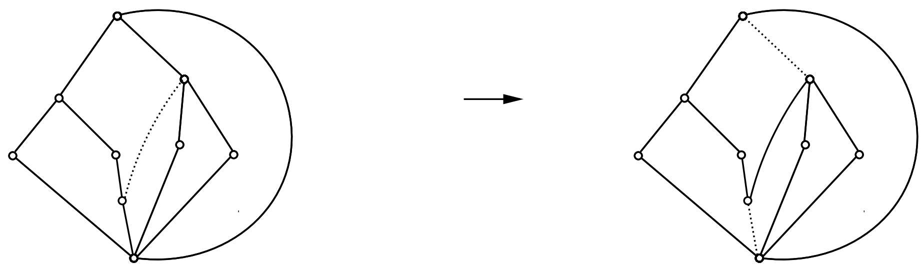

Figure 2: An example of an improvement operation of Type II

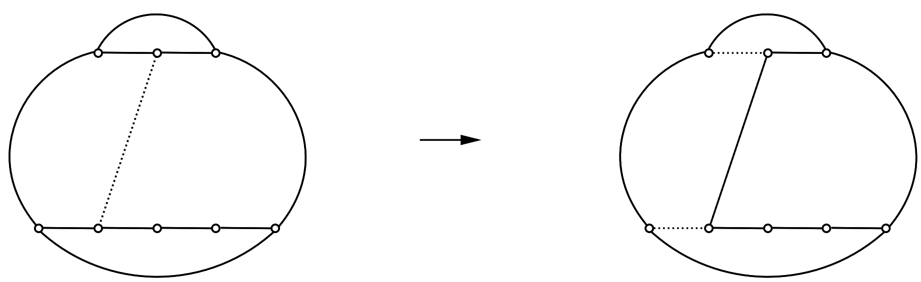

Figure 3: An example of an improvement operation of Type III

exist on F. If it does, it calls the procedure described above for S and u *recursively* on the internal vertices of the newly appearing strong short segments provided that no improvement process has been previously called on a given segment and an internal vertex. These calls are performed for all H on u and for each internal vertex $u \in S$. If there is an improvement operation in one of the recursive calls, the called function returns and the caller performs a specific *deletion operation* as follows. It attempts to delete the edges from $F \cup H$ in the order F, H, where the order within the sets F and H are immaterial. Specifically, it deletes an edge as long as the residual graph remains feasible. This enforces to keep the edges in H in the solution. Examples of these operations are

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Algorithm 1: 2-VCSS($G(V, E)$)

Let F be an inclusion-wise minimal 2-VCSS of G

while *there is a strong short segment S on F an an internal vertex u of S on which an improvement process has not been called* **do**

  IMPROVEMENT-PROCESS(F, S, u)

return F

Algorithm 2: IMPROVEMENT-PROCESS(F, S, u)

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if there is an improvement operation that can be performed on u then
    Apply the improvement operation on F
    return
for each set of critical edges H on u do
    Let 𝒮 be the set of strong short segments on F ∪ H that do not exist on F
    for each strong short segment T in 𝒮 and an internal vertex v of T do
        if no improvement process has been called on (T, v) then
            IMPROVEMENT-PROCESS(F ∪ H, T, v)
            if there is an improvement operation performed in
              IMPROVEMENT-PROCESS(F ∪ H, T, v) then
                Perform deletion operation on F ∪ H in the order F, H
                return
if there is no improvement operation performed in any of the recursive calls above then
    Restore F to the original set considered before the function call
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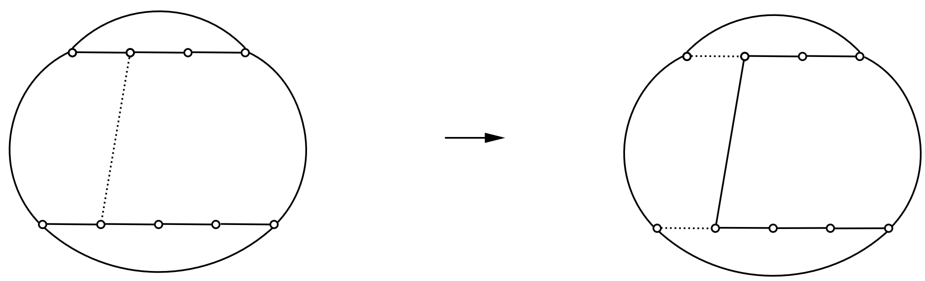


Figure 4: An example of an improvement operation of Type IV

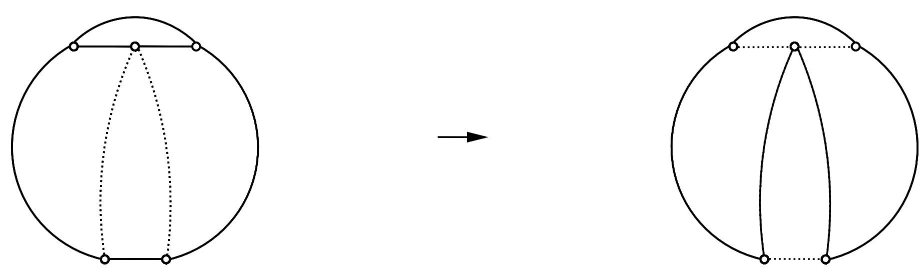


Figure 5: An example of an improvement operation of Type V

given in Figure 6-Figure 8, where the depth of the recursion tree is 2. After the reverse-delete operation, the current function call returns. If after all the recursive calls from u there is no improvement operation performed, the solution F is restored back to the original one before the function call on u. The main iterations continue until there is no S and u on which we can perform an improvement process.

This completes the definition of the algorithm for 2-VCSS. For 2-ECSS, one final step is performed, which excludes all the redundant trivial segments, i.e., the trivial segments whose removal does not violate 2-edge-connectivity.

Proposition 1. *Algorithm 1 runs in polynomial time.*

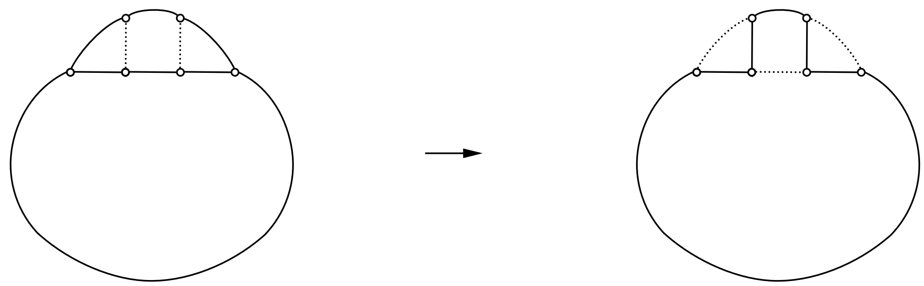


Figure 6: An example of an improvement process of recursion depth 2

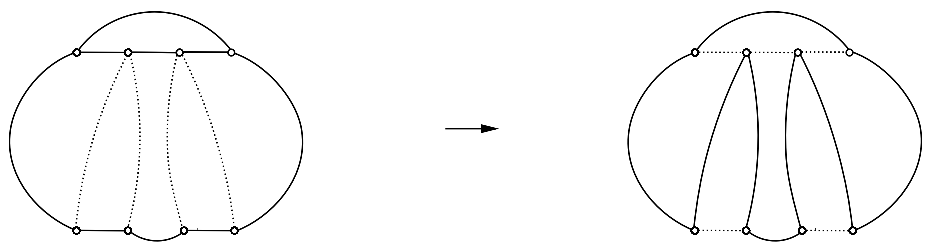


Figure 7: An example of an improvement process of recursion depth 2

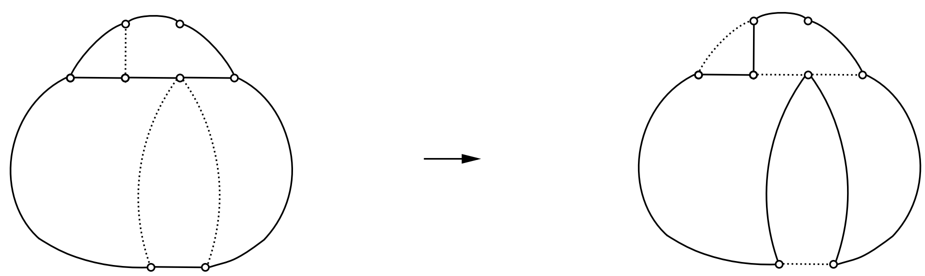


Figure 8: An example of an improvement process of recursion depth 2

Proof. It suffices to see that the main loop of IMPROVEMENT-PROCESS terminates in polynomial number of steps. As noted, there are polynomially many sets H, since $|H|$ is constant. Starting from an internal vertex u of a strong short segment S, consider the recursion tree in which each node represents a recursive function call. By definition, each node of this tree is associated to an internal vertex of a strong short segment that does not exist in any other node in the tree. This implies that the number of nodes in the tree is $O(|V|)$. So the algorithm terminates in polynomial number of steps. □

4 Proof of the Approximation Factor 4/3 for 2-ECSS and 2-VCSS

Let F be a solution returned by Algorithm 1. Let $opt(G)$ denote the value of an optimal 2-ECSS or 2-VCSS on G, depending on the type of problem solved.

Proposition 2. *Let S_1 and S_2 be two distinct strong short segments on F. Then there are no edges between an internal vertex of S_1 and an internal vertex of S_2.*

Proof. Existence of such an edge contradicts an improvement operation. □

Lemma 3. *Let u be an internal vertex of a strong short segment S on F. Then there exists an optimal solution O on G such that O contains at most 2 edges that are in $E \setminus F$ and incident to u.*

Proof. It is clear that the number of such edges might be 2. Let them be e_1 and e_2. Suppose O contains a third such edge $e_3 \in E \setminus F$. Then by the structure of a 2-ECSS, one of e_1, e_2, e_3, say $e = (u, v)$, satisfies the following. Take a neighbor w of v such that $f = (v, w) \in F$. There is another optimal solution $O' = O \cup \{f\} \setminus \{e\}$. Note next that by Proposition 2, the vertex v cannot be an internal vertex of a strong short segment. This completes the proof. □

Lemma 4. *There exists G_1 and a feasible $F_1 \subseteq E(G_1)$ such that the following hold:*

1. *For any internal vertex s of a strong short segment on F_1, there is no $t \in V$ with $(s,t) \in E(G_1) \setminus F_1$.*
2. *F_1 is minimal with respect to inclusion.*
3. $\frac{|F_1|}{opt(G_1)} \leq \frac{4}{3} \Rightarrow \frac{|F|}{opt(G)} \leq \frac{4}{3}$.

Proof. We reduce G to G_1 and F to F_1 by performing a series of operations. Let O be an optimal solution on G implied by Lemma 3, and let S be a strong short segment on F. Let $O(S)$ be the set of edges in this solution incident to the internal vertices of S. Let $F' = F \cup O(S) \setminus P$, where P is the edge set of the redundant trivial segments on $F \cup O(S)$ such that $P \subseteq F \setminus O(S)$. Examples of this operation are illustrated in Figure 9 and Figure 10. Let $E(S)$ denote the set of edges incident to the internal vertices of S on $E(G)$, which excludes the edges in F'. Delete the edges in $E(S)$ from G to obtain G'. Perform these operations recursively on the new strong short segments that appear on F', which we call *emerging segments*. By Proposition 2, the recursion terminates. After the recursion starting from S terminates, continue performing the described operations on the strong short segments on the residual solution and the graph. Let the results be F_1 and G_1. Given these, the first claim of the lemma holds, since there is no edge in $E(G_1) \setminus F_1$ incident to the internal vertices of a strong short segment on F_1 by construction. The second claim of the lemma holds,

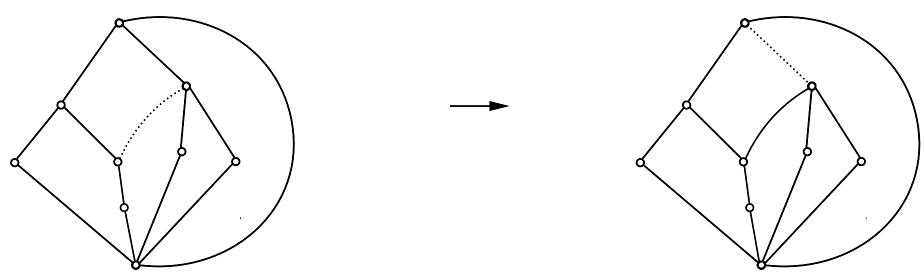


Figure 9: A transition from F to F_1 on a 2-VCSS with $|O(S)| = 1$

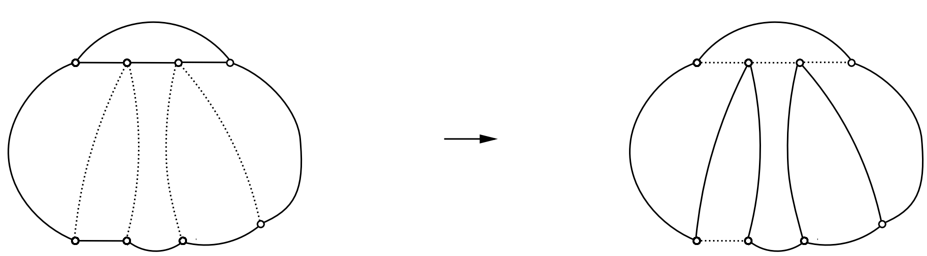


Figure 10: A transition from F to F_1 on a 2-ECSS with $|O(S)| = 4$

since there are no redundant trivial segments on F_1 by the described operations. We now show that the third claim also holds.

Claim 5. $|F_1| \geq |F|$.

Proof. Let S be a strong short segment on which we start the recursive operations above or an emerging segment. It suffices to see that $|P| \leq |O(S)|$. By the algorithm and the construction of F_1, there is no improvement process performed on S that has improved the cost of the solution. In this case $|P| > |O(S)|$ derives a contradiction. In particular by all the listed improvement operations, we cannot have the configurations on the left hand sides of Figure 1-Figure 8. □

Claim 6. $opt(G_1) \leq opt(G)$.

Proof. This follows from the fact that for any strong short segment S on F_1, $E(S)$ does not contain any edge from O. □

Combining Claim 5 and Claim 6 we obtain the third claim of the lemma, which completes the proof. □

Let G_1 and F_1 be as implied by Lemma 4.

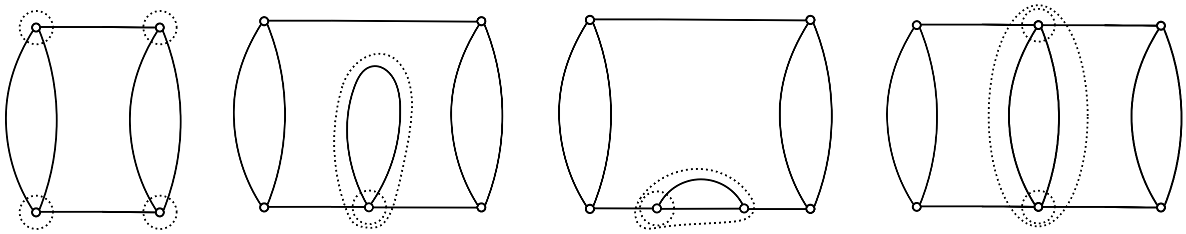
Lemma 7.

$$\frac{|F_1|}{opt(G_1)} \leq \frac{4}{3}.$$

Proof. We construct a feasible dual solution in (EC-D) with total value at least $\frac{3}{4}|F_1|$. Recall that there are no strong long segments on F_1. Given a strong short segment S and an internal vertex s of S, assign $y_{\{s\}} = 3/4$. If the segment is a 3-segment, let $z_e = 1/2$ for the middle edge e of the segment to maintain feasibility. Note that the overall assignment is feasible by the first claim of Lemma 4.

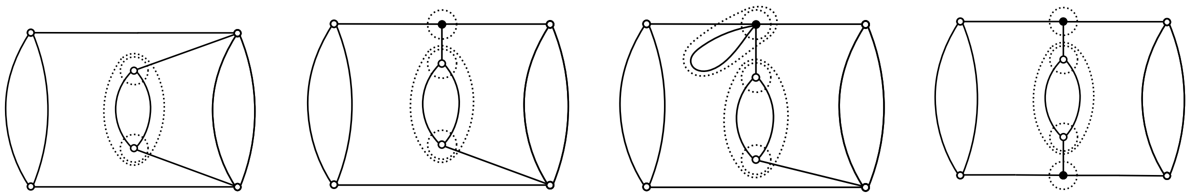
We distinguish a dual value we assign and its contribution in the objective function of (EC-D), which is twice the dual value. The latter is called the *dual contribution*, and the associated dual is said to *contribute* a certain value. We continue with a cost sharing argument as follows. The cost of each segment will be countered with a unique set of dual contributions with ratio at least $\frac{3}{4}$, which establishes the main result. To this aim, we impose that the strong short segments are paid by the dual contributions defined on their internal vertices and edges. This is with ratio at least $\frac{3}{4}$, since for a 2-segment the total dual contribution is $\frac{3}{2}$, and for a 3-segment the total dual contribution is $\frac{3}{2} + \frac{3}{2} - \frac{1}{2} = \frac{5}{2}$. The cost of the strong long segments are paid by the dual contributions defined on their internal vertices, which is with ratio at least $\frac{3}{4}$, since the cost of a long segment is at least 4. There remains the cost of weak segments. The cost of a weak segment is countered with the dual contributions of its internal vertices and a new set of contributions y we will later define. Thus for a weak ℓ-segment, the total contribution is $\ell - 1 + y$. We impose that $\ell - 1$ of this optimally pays for the $\ell - 1$ edges of the segment, and show $y \geq 3/4$. In general, we construct a total value of at least $\frac{3k}{4}$, where k is the total number of weak segments.

We first show this for a 2-ECSS by an induction on k. We consider the base case $k = 2$ in which the two weak segments do not share a common end vertex. This is depicted in Figure 11a. In this case assign $y_{\{v\}} = 1/4$ for each end vertex v of a weak segment, so that the total dual contribution defined for the weak segments is 2, which satisfies the result. In the inductive step one may introduce one, two, three, or four new weak segments by extending the graph in the induction hypothesis. These are given in Figure 11 and Figure 12, where we depict the extending subgraphs



(a) (b) (c) (d)

Figure 11



(a) (b) (c) (d)

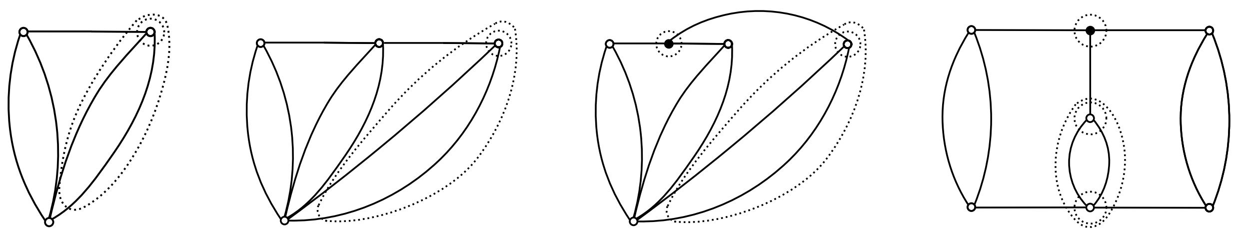
Figure 12

in their simplest form. We assume a dual assignment on the internal vertices of the segments in the extending subgraph as described in the first paragraph, so that all its segments are covered with ratio at least $\frac{3}{4}$. We then describe a further dual assignment to cover the newly introduced weak segments.

In Figure 11b and Figure 11c one new weak segment is introduced. Let u be a newly introduced high-degree vertex. If there is no strong short segment with an end vertex u, we define $y_{\{u\}} = 1/2$. Otherwise, we define $y_{\{u\}} = 1/4$ and $y_S = 1/4$, where S is the set of vertices of strong short segments with an end vertex u. Note that this is feasible by the first claim of Lemma 4. These duals are shown in the figures with dotted ovals. In either case the dual contribution is 1, which optimally covers the new weak segment.

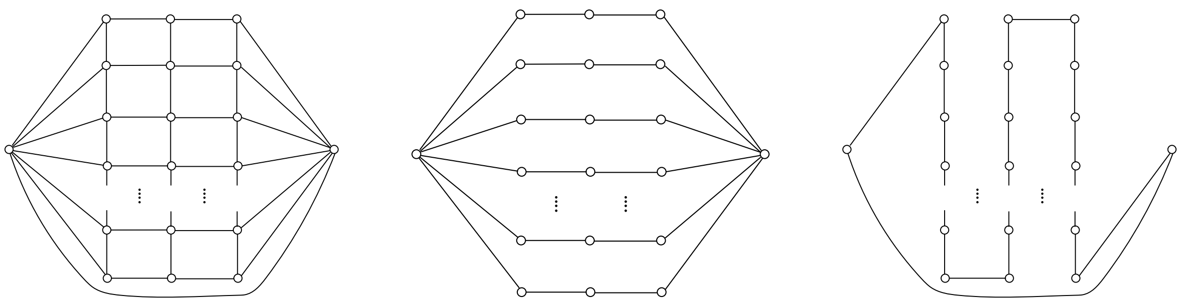
In Figure 11d two new weak segments are introduced. Let u and v be two newly introduced high-degree vertices, which are also end vertices of weak segments. If u and v are the end vertices of strong short segments, assign $y_{\{u\}} = y_{\{v\}} = 1/4$, and $y_S = 1/4$, where S is the set of vertices of strong short segments with an end vertex u. These duals are also shown via dotted ovals in the figure. Otherwise, e.g., u is not shared by a strong short segment, let $y_{\{u\}} = 1/2$. The total dual contribution by u and v together with y_S is at least $3/2$ in both cases, covering the two new weak segments with ratio at least $\frac{3}{4}$. We do not depict the generalization of Figure 11d, analogous to the one from Figure 11b to Figure 11c, which does not change the analysis.

The argument for the configuration in Figure 12a is identical to that of Figure 11d. In Figure 12b and Figure 12c three new weak segments are introduced. If an end vertex is only incident to weak segments as in Figure 12b, it is assigned the dual value $1/2$ and hence contributes 1 (See the vertex depicted as a black dot in the figure). Otherwise, we define dual values around it as described for Figure 11b, which again contributes 1. This is depicted in Figure 12c. In both cases the other vertices contribute $3/2$, as described for Figure 11d, thus summing up to at least $5/2$. This covers the three newly introduced weak segments with ratio at least $\frac{5}{6}$. Figure 12d is a straightforward generalization of Figure 12b. The total new dual contribution is thus at least $\frac{7}{2}$, which covers the four new weak segments with ratio at least $\frac{7}{8}$. We do not depict the generalizations of Figure 12d,



(a) **(b)** **(c)** **(d)**

Figure 13



(a) **(b)** **(c)**

Figure 14: A tight example for the algorithm: (a) Input graph; (b) A solution returned by the algorithm; (c) An optimal solution.

analogous to the one from Figure 12b to Figure 12c, which does not change the analysis. This completes the induction.

We now show the same result for a 2-VCSS. By definition, both the base case and the inductive step for a 2-ECSS are subsets of those for a 2-VCSS except the ones depicted in Figure 11b and Figure 12c. In addition, a 2-VCSS might have one trivial segment in the base case as depicted in Figure 13a. Let $e \in F_1$ be its edge, and fix an end vertex u of the segment. By the second constraint in (VC), there exists a high-degree vertex v and $W = (W_t, W_h)$ such that $V = \{v\} \cup W_t \cup W_h$, $u \in W_t$, and e covers W. If u is not shared by any strong short segment, define $y_{\{u\}} = 1/2$, which optimally covers e. Otherwise, define $y_{\{u\}} = 1/4$ and $y_S = 1/4$, where S is the set of vertices of strong short segments with an end vertex u, but excluding the vertex v. Note that this is feasible by the first claim of Lemma 4. These duals are shown as dotted ovals in the figure. The total value contributed by these duals is again 1, which optimally covers e.

The inductive step might introduce one, two, or three new weak segments as depicted in Figure 13b, Figure 13c, and Figure 13d, which are configurations that are not covered by the previous ones considered for a 2-ECSS. In Figure 13b we define a dual value of 1/4 on the newly introduced end vertex of the weak segment. We also define the value 1/4 for the dual, which is defined as in the base case. These duals in total contribute 1, and hence optimally cover the new trivial segment. Figure 13c additionally contains a vertex incident to only weak segments on which we define a dual of value 1/2, so that the two new weak segments are optimally covered. The analysis of the configuration in Figure 13d is identical to that of Figure 12b. This completes the induction for a 2-VCSS, and the proof. □

Theorem 8.
$$|F| \leq \frac{4}{3} opt(G).$$

Proof. Follows from Lemma 7 and Lemma 4. □

5 A Tight Example

A tight example for the algorithm is given in Figure 14. The solution returned by the algorithm has cost $4k$. The optimal solution has cost $3k + 2$.

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