A Unified Approach for Approximating 2-Edge-Connected Spanning Subgraph and 2-Vertex-Connected Spanning Subgraph

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Abstract

We provide algorithms for the minimum 2-edge-connected spanning subgraph problem and the minimum 2-vertex-connected spanning subgraph problem with approximation ratio $\frac{4}{3}$.

1 Introduction

We consider two fundamental connectivity problems, namely the minimum 2-edge-connected spanning subgraph problem (2-ECSS) and the minimum 2-vertex-connected spanning subgraph problem (2-VCSS).

In 2-ECSS, given an undirected simple graph G = (V, E), one finds a 2-edge-connected spanning subgraph (2-ECSS) of G with minimum number of edges. The problem remains NP-hard and MAX SNP-hard even for subcubic graphs [6]. The first result improving the approximation factor 2 came from Khuller and Vishkin [14], which is a $\frac{3}{2}$ -approximation algorithm. Cheriyan, Sebö and Szigeti [2] improved the factor to $\frac{17}{12}$. Vempala and Vetta [18], and Jothi, Raghavachari and Varadarajan [13] claimed to have $\frac{4}{3}$ and $\frac{5}{4}$ approximations, respectively. Krysta and Kumar [15] went on to give a $(\frac{4}{3} - \epsilon)$ -approximation for some small $\epsilon > 0$ assuming the result of Vempala and Vetta [18]. A relatively recent paper by Sebö and Vygen [17] provides a $\frac{4}{3}$ -approximation algorithm by using ear decompositions, and mentions that the aforementioned claimed approximation ratio $\frac{5}{4}$ has not appeared with a complete proof in a fully refereed publication. The result of Vempala and Vetta [18], which was initially incomplete, recently re-appeared in [12]. Given this, it is not clear if the ratio $(\frac{4}{3} - \epsilon)$ by Krysta and Kumar [15] still holds. Very recently, Garg, Grandoni, and Ameli [8] improved upon the ratio $\frac{4}{3}$ by a constant $\frac{1}{130} > \epsilon > \frac{1}{140}$, which stands as the best current result for approximating the problem.

In 2-VCSS, given an undirected simple graph G = (V, E), one finds a 2-vertex-connected (or simply 2-connected) spanning subgraph (2-VCSS) of G with minimum number of edges. This problem is NP-hard via a reduction from the Hamiltonian cycle problem. Furthermore, by the result of Czumaj and Lingas [7], it does not admit a PTAS unless P = NP. The first result improving the factor 2 for 2-VCSS came from Khuller and Vishkin [14], which is a $\frac{5}{3}$ -approximation algorithm. Garg, Vempala, and Singla [9] improved the approximation ratio to $\frac{3}{2}$. Cheriyan and Thurimella [3] also provides the same approximation ratio in a more general context, where they consider k-connectivity. Vempala and Vetta [18] claimed the ratio $\frac{4}{3}$, which is shown to be not valid by Heeger and Vygen [11]. Jothi, Raghavachari and Varadarajan [13] claimed the ratio $\frac{5}{4}$.

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However, this claim has been later withdrawn (see [10]). Gubbala and Raghavachari [10] claimed to have a $\frac{9}{7}$ -approximation algorithm. The only complete (and exceedingly long) proof of this claim is in Gubbala's thesis [16], which has not appeared anywhere else. To the best of our knowledge, the algorithm by Heeger and Vygen [11] with factor $\frac{10}{7}$ stands as the first refereed improvement over the factor $\frac{3}{2}$ after a long hiatus. This paper also contains a somewhat more detailed discussion about the aforementioned claimed results, implying that theirs is the first improvement. Very recently, Bosch-Calvo, Grandoni, and Ameli provided an algorithm with approximation ratio $\frac{4}{3}$ [1], the current best factor.

Although the two problems we consider are closely related to each other, as noted above, the work on these problems have generally appeared in distinct publications. The reason is that the solution being 2-edge-connected or 2-connected enforces different approaches if the algorithmic ideas are tightly coupled with the structure of the solution. We give a unified strategy to solve these problems, where we use an algorithmic idea almost oblivious to whether the solution should be 2-edge-connected or 2-connected. The starting point is [5] in which we gave an algorithm for 2-ECSS. Unfortunately, its analysis therein is not clear, and contains several inconsistencies, even with the corrigendum published after the paper [4]. The purpose of this paper is twofold: We provide a new algorithm for 2-ECSS, which starts from the ideas employed in [5], but with a clear analysis, thereby obtaining the ratio $\frac{4}{3}$. Secondly, we show that the same algorithmic idea can be applied to 2-VCSS to attain the same approximation ratio $\frac{4}{3}$, matching the current best approximation ratio. Both the algorithms and their analyses are considerably simpler compared to previous approaches. In essence, we do not use anything but a recursive local search over an inclusion-wise minimal starting solution.

2 Preliminary Definitions

For both of the problems, we will use the lower bound derived from the dual of the natural LP relaxation for 2-ECSS. Here, $\delta(S)$ denotes the set of edges with one end in the cut S and the other not in S.

$$\begin{array}{ll} \text{minimize} & \sum_{e \in E} x_e & (\text{EC}) \\ \text{subject to} & \sum_{e \in \delta(S)} x_e \geq 2, \qquad \forall \emptyset \subset S \subset V, \\ & 1 \geq x_e \geq 0, \qquad \forall e \in E. \end{array}$$

The following is the dual of (EC).

$$\begin{array}{ll} \text{maximize} & \sum_{\emptyset \subset S \subset V} 2y_S - \sum_{e \in E} z_e & (\text{EC-D}) \\ \text{subject to} & \sum_{S:e \in \delta(S)} y_S \leq 1 + z_e, & \forall e \in E, \\ & y_S \geq 0, & \forall \emptyset \subset S \subset V, \\ & z_e \geq 0, & \forall e \in E. \end{array}$$

We assume that the input graph G is 2-connected. Otherwise, the algorithm of the next section can be executed on blocks (maximal 2-connected subgraphs) of G separately. This is without loss of generality, since in that case the value of an optimal solution for 2-ECSS is the sum of those of blocks, and we can argue the approximation ratio only within a block. Given a vertex $v \in V$ and a feasible solution F, if the degree of v in the graph (V, F) is at least 3, it is called a *high-degree vertex* on F. For a path $P = v_1v_2 \dots v_{k-1}v_k$, v_1 and v_k are the *end vertices* of P, and all the other vertices are the *internal vertices* of P. A path whose internal vertices have degree 2 in the graph (V, F) is called a *plain path* on F. A maximal plain path is called a *segment*. The length of a segment is the number of edges on the segment. If the length of a segment is ℓ , it is called an ℓ -segment. A 1-segment is also called a *trivial segment*. If a 2-ECSS remains feasible upon removal of an edge, the edge is called *redundant*. An ℓ -segment with $\ell \geq 2$ is called a *short segment* if $\ell \leq 3$, otherwise a *long segment*. Given a 2-VCSS F, if the removal of a segment from F violates 2-connectivity, it is called a *weak segment* on F, otherwise a *strong segment* on F. If the end vertices of a segment (on a 2-ECSS) are identical, it is called a *closed segment*.

3 The Algorithm for 2-ECSS and 2-VCSS

The algorithm starts by considering an inclusion-wise minimal 2-VCSS on G (Recall that we assume G is 2-connected). Let F be such a solution. Given an internal vertex u of a strong short segment S on F, let $N_E(u)$ denote the set of edges incident to u in E. The algorithm modifies F via what we call *improvement processes*. An improvement process first tries to replace F by $(F \setminus B) \cup A$ while maintaining feasibility, where $A \subseteq E \setminus F$ (the replacing edge set) is a set of k edges called a *critical edge set*, and $B \subset F$ (the replaced edge set) is a set of k + 1 edges, $1 \leq k \leq 2$. Specifically, we seek such A to be a subset of $N_{E \setminus F}(u)$ for an internal vertex u of some strong short segment S on F. If there is such A, the described operation is called an *improvement operation*. Examples of improvement operations are given in Figure 1 and Figure 2.

If there is no improvement operation that can be performed on u, fix a critical edge set A, and let \mathcal{A}_F be the set of strong short segments on $F \cup A$ that do not exist on F. Let v be an internal vertex of some segment $S \in \mathcal{A}_F$. The algorithm calls the procedure described above on v recursively provided that no improvement process has been previously called on v. These calls are performed for all A on u and for each internal vertex $u \in S$. If there is an improvement operation in one of these calls, the called function returns and the caller performs a specific deletion operation as follows. It attempts to delete the edges from $F \cup A$ in the order F, A, where the order within the sets F and A are immaterial. Specifically, it deletes an edge as long as the residual graph remains a 2-VCSS. This enforces to keep the edges in A in the solution. Examples of these operations are

Algorithm 1: 2-ECSS-2-VCSS(G(V, E))

// Initialization

Let F be an inclusion-wise minimal 2-VCSS of G

// Improvement processes

while there is a strong short segment S on F and an internal vertex u of S on which an improvement process has not been called **do**

IMPROVEMENT-PROCESS(F, S, u)

- // Clean-up for 2-ECSS
- if computing a 2-ECSS then

Delete redundant edges from F to obtain \overline{F}

return (F, \overline{F})

Algorithm	2:	IMPROVEMENT-PROCESS	(F)	, S	u)
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if there is an improvement operation that can be performed on u then Apply an improvement operation on u return				
for each critical edge set A incident to u do $ $ Let \mathcal{A}_F be the set of strong short segments on $F \cup A$ that do not exist on F				
for each segment T in \mathcal{A}_F and each internal vertex v of T do				
if no improvement process has been called on (T, v) then				
$ $ Improvement-Process $(F \cup A, T, v)$				
if there is an improvement operation performed in				
IMPROVEMENT-PROCESS $(F \cup A, T, v)$ then				
Perform deletion operation on $F \cup A$ in the order F, A				
return				
if there is no improvement operation performed in any of the calls above then				

Restore F to the original set considered before the function call



Figure 1: An example of an improvement operation, where the critical edge set is shown via dotted lines on the left-hand side



Figure 2: An example of an improvement operation, where the critical edge set is shown via dotted lines on the left-hand side

given in Figure 3 and Figure 4, where the depth of the recursion tree is 2. After the reverse-delete operation, the current function call returns. If no improvement operation is performed in any of the calls, the solution F is restored to the original one before the function call on u. The iterations continue until there is no strong short segment S and u on which we can perform an improvement process.

This completes the definition of the algorithm for 2-VCSS. For 2-ECSS, one final step is performed, which excludes all the redundant edges, i.e., the trivial segments whose removal does not violate 2-edge-connectivity. The result of this operation is \overline{F} .

Proposition 1. Algorithm 1 can be implemented in polynomial-time.



Figure 3: An example of an improvement process of recursion depth 2, where the edges in the sets A considered by the algorithm is shown via dotted lines on the left-hand side



Figure 4: An example of an improvement process of recursion depth 2, where the edges in the sets A considered by the algorithm is shown via dotted lines on the left-hand side

Proof. It suffices to see that the main loop of IMPROVEMENT-PROCESS terminates in polynomial number of operations. Note first that there are polynomially many sets A considered for a single vertex u, since |A| is constant. Starting from an internal vertex u of a strong short segment S, consider the recursion tree in which each node represents a recursive function call. By definition, each node of this tree is associated to an internal vertex of a strong short segment. A vertex can be an internal vertex of a constant number of strong short segments. Since an improvement process is called at most once on a strong short segment-internal vertex pair, this implies that the number of nodes in the tree is polynomially bounded. So the algorithm terminates in polynomial number of operations.

4 Proof of the Approximation Factor 4/3 for 2-ECSS and 2-VCSS

Let (F, \overline{F}) be a solution returned by Algorithm 1, and opt(G) denote the value of an optimal 2-ECSS on G.

Lemma 2. A short segment S on \overline{F} is not a closed segment.

Proof. Recall that the input graph is assumed to be 2-connected. This implies that if S is a closed segment with an internal vertex u and the end vertex v, then there exists $w \in V$ with $(u, w) \in E \setminus F$. In this case however, the algorithm includes such an edge into the solution instead of (u, v), which is a contradiction.

Lemma 3. There exists a graph G_1 (depending on G), a 2-VCSS $F_1 \subseteq E(G_1)$, and a 2-ECSS $F_2 \subseteq E(G_1)$ such that the following hold.

1. F_1 is minimal with respect to inclusion.

- 2. F_2 is obtained from F_1 by deleting redundant edges.
- 3. For any internal vertex s of a strong short segment S on F_1 , there is no edge $e \in E(G_1) \setminus F_1$ incident to s.
- $4. \quad \frac{|F_1|}{opt(G_1)} \leq \frac{4}{3} \Rightarrow \frac{|F|}{opt(G)} \leq \frac{4}{3}. \quad \frac{|F_2|}{opt(G_1)} \leq \frac{4}{3} \Rightarrow \frac{|\overline{F}|}{opt(G)} \leq \frac{4}{3}.$

Proof. We reduce G to G_1 and F to F_1 by performing a series of operations. Let S be a strong short segment on F, and s be an internal vertex of S. Let O be an optimal 2-ECSS on G. Clearly, O contains two edges incident to s, say e_1 and e_2 . Assume it contains a third edge e_3 incident to s. Let the other end vertices of these edges be w_1, w_2 , and w_3 , respectively. By the improvement operations performed by the algorithm, none of these vertices is an internal vertex of a strong short segment. If O contains all the edges incident to w_i that are in F, we call w_i a special vertex, for i = 1, 2, 3.

Claim 4. There is at most one special vertex in the set $\{w_1, w_2, w_3\}$.

Proof. Assume without loss of generality that w_1 and w_2 are special vertices. Then by the structure of a 2-ECSS, we can discard e_1 or e_2 from O without violating feasibility, which contradicts its optimality.

Claim 5. There exists an optimal 2-ECSS O' on G such that O' contains 2 edges incident to s.

Proof. By Claim 4, there are at least two vertices in the set $\{w_1, w_2, w_3\}$ that are not special. Let two of them be without loss of generality w_2 and w_3 . By the structure of a 2-ECSS, one of these vertices, say w_2 , satisfies the following. There is a neighbor w'_2 of w_2 such that $f = (w_2, w'_2) \in F \setminus O$, and $O' = O \cup \{f\} \setminus \{e_2\}$ is another optimal solution. In this case the degree of s on O' is 2, which completes the proof.

Let O'(S) be the set of edges in this solution incident to the internal vertices of S, and let $F' = F \cup O'(S) \setminus P$ be a minimal 2-VCSS on G, where $P \subseteq F \setminus O'(S)$. Examples of this operation are illustrated in Figure 5 and Figure 6. Let E(S) denote the set of edges incident to the internal vertices of S in $E(G) \setminus O'(S)$. Note that this set contains P. Delete the edges in E(S) from G to obtain G'. Switch to an optimal solution on G' implied by Claim 5. We call the new strong short segments that appear on F' the emerging segments. Apply all these operations recursively on the emerging segments. Note that since none of the aforementioned vertices w_1, w_2 , and w_3 can be an internal vertex of a strong short segment as noted, the switch from O to O' cannot be reversed. After the recursion starting from S terminates, continue performing the described operations on the strong short segments on the residual solution and the graph. Let the results be F_1 and G_1 . Given these, the first claim of the lemma follows from Claim 5. The third claim of the lemma holds, since there is no edge in $E(G_1) \setminus F_1$ incident to the internal vertices of a strong short segment on F_1 by construction. Let F_2 be a 2-ECSS obtained by deleting redundant edges from F_1 , thus satisfying the second claim of the lemma. We now show that the fourth claim holds.

Claim 6. $|F_1| \ge |F|$. $|F_2| \ge |\overline{F}|$.

Proof. Let S be a strong short segment on which we start the recursive operations above or an emerging segment. For $|F_1| \ge |F|$, it suffices to see that $|P| \le |O'(S)|$ in the construction described above. Note that by the algorithm and the construction of F_1 , there is no improvement process performed on S that has improved the cost of the solution. Given this, |P| > |O'(S)| derives a contradiction. In particular, we cannot have the configurations on the left hand sides of Figure 1-Figure 4. Note next that if e is an edge deleted from F_1 to obtain F_2 , then we also have that



Figure 5: A transition from F to F_1 with |P| = 1, where the edges in O'(S) are shown via dotted lines on the left-hand side, and the edges in P are shown via dotted lines on the right-hand side



Figure 6: A transition from F to F_1 with |P| = 4, where the edges in O'(S) are shown via dotted lines on the left-hand side, and the edges in P are shown via dotted lines on the right-hand side

e belongs to the set of redundant edges deleted to obtain the 2-ECSS \overline{F} by the algorithm in the last step. Given this, the first assertion that $|F_1| \ge |F|$ implies $|F_2| \ge |\overline{F}|$. This completes the proof.

We next note that $opt(G_1) \leq opt(G)$. This follows from our construction ensuring that there is an optimal solution O such that for any strong short segment S on F_1 , E(S) does not contain any edge from O. Combining this with Claim 6, we obtain $\frac{|F|}{opt(G)} \leq \frac{|F_1|}{opt(G_1)}$ and $\frac{|\overline{F}|}{opt(G_1)} \leq \frac{|F_2|}{opt(G_1)}$, which establishes the fourth claim of the lemma and completes the proof.

Let G_1 , F_1 , and F_2 be as implied by Lemma 3.

Lemma 7.
$$\frac{|F_1|}{opt(G_1)} \leq \frac{4}{3}$$
. $\frac{|F_2|}{opt(G_1)} \leq \frac{4}{3}$.

Proof. We first construct a feasible dual solution in (EC-D) with total value at least $\frac{3}{4}|F_2|$, and later establish a construction with value at least $\frac{3}{4}|F_1|$, considering the cases specific to a 2-VCSS. Given a strong short segment S on F_1 and an internal vertex s of S, assign $y_{\{s\}} = 3/4$. If the segment is a 3-segment, let $z_e = 1/2$ for the middle edge e of the segment to maintain feasibility. Let $y_{\{w\}} = 1/2$ for the internal vertices w of strong long segments and weak segments on F_1 . Note that the overall assignment is feasible by the third claim of Lemma 3.

We distinguish a dual value we assign and its contribution in the objective function of (EC-D), which is twice the dual value. The latter is called the *dual contribution*, and the associated dual is said to *contribute* a certain value. We use a cost sharing argument, so that the cost of a set of segments is countered with a unique set of dual contributions with ratio at least $\frac{3}{4}$, which establishes the main result. To this aim, we impose that the strong short segments are paid by the dual contributions defined on their internal vertices and edges. This is with ratio at least $\frac{3}{4}$, since for a 2-segment the total dual contribution is $\frac{3}{2}$, and for a 3-segment the total dual contribution is $\frac{3}{2} + \frac{3}{2} - \frac{1}{2} = \frac{5}{2}$. The cost of the strong long segments are paid by the dual contributions defined on



their internal vertices, which is with ratio at least $\frac{3}{4}$, since the cost of a long segment is at least 4. The cost of a weak segment is countered with the dual contributions of its internal vertices and a new set of contributions y we will later define. Thus for a weak ℓ -segment, the total contribution is $\ell - 1 + y$. We impose that $\ell - 1$ of this optimally pays for the $\ell - 1$ edges of the segment, and show $y \geq 3/4$. Let k be the number of weak segments on F_1 , excluding the trivial segments defined by the redundant edges. We construct a total value of at least $\frac{3k}{4}$, where k is the total number of weak segments.

We argue by induction on k. We first consider the base case k = 1, where the weak segment is not a trivial segment. This is depicted in Figure 7a. Let u be an end vertex of this weak segment. If u is not shared by any strong short segment, define $y_{\{u\}} = 1/2$, which optimally covers the weak segment. Otherwise, there exists a high-degree vertex v such that u and v are the end vertices of a strong short segment. In this case define $y_{\{u\}} = 1/4$ and $y_S = 1/4$, where S is the set of vertices of strong short segments with an end vertex u, but excluding the vertex v. Note that this is feasible by the third claim of Lemma 3. These duals are shown as dotted ovals in the figure. The total value contributed by these duals is again 1, which optimally covers e. The inductive step might introduce one or two new weak segments as depicted in Figure 7b and Figure 7c. The analysis of the configuration in Figure 7b is identical to that of the base case. Figure 7c additionally contains a vertex incident to only weak segments on which we define a dual of value 1/2, so that the two new weak segments are optimally covered.

We now consider the base case k = 2 in which the two weak segments do not share a common end vertex. This is depicted in Figure 8a. In this case assign $y_{\{v\}} = 1/4$ for each end vertex vof a weak segment, so that the total dual contribution defined for the weak segments is 2, which satisfies the result. In the inductive step one may introduce one, two, three, or four new weak segments by extending the graph in the induction hypothesis. These are given Figure 8, Figure 9, and Figure 7d, where we depict the extending subgraphs in their simplest form. In what follows, we tacitly assume the third claim of Lemma 3, which forbid the existence of edges that would violate the feasibility of the dual solution.

In Figure 8b and Figure 8c one new weak segment is introduced. Let u be a newly introduced high-degree vertex. If there is no strong short segment with an end vertex u, we define $y_{\{u\}} = 1/2$. Otherwise, we define $y_{\{u\}} = 1/4$ and $y_S = 1/4$, where S is the set of vertices of strong short segments with an end vertex u. This is feasible by the third claim of Lemma 3, a fact that will also be used in the following cases. The duals are shown in the figures with dotted ovals. Note that the closed segment in Figure 8b cannot be a short segment by Lemma 2. In either case the dual contribution is 1, which optimally covers the new weak segment.

In Figure 8d two new weak segments are introduced. Let u and v be two newly introduced high-degree vertices, which are also the end vertices of the weak segments. If u and v are the end vertices of strong short segments, assign $y_{\{u\}} = y_{\{v\}} = 1/4$, and $y_S = 1/4$, where S is the set of



vertices of strong short segments with an end vertex u. Otherwise, let $y_{\{u\}} = 1/2$. The total dual contribution by u and v together with y_S is at least 3/2 in both cases, covering the two new weak segments with ratio at least $\frac{3}{4}$. We do not depict the generalization of Figure 8d, analogous to the one from Figure 8b to Figure 8c, which does not change the analysis.

The argument for the configuration in Figure 9a is identical to that of Figure 8d. In Figure 9b, Figure 9c, and Figure 7d (where one of the newly introduced weak segments is not a trivial segment) three new weak segments are introduced. If an end vertex is only incident to weak segments, it is assigned the dual value 1/2 and hence contributes 1 (See the vertex depicted as a black dot in the figures). Otherwise, we define dual values around it as described for Figure 8b, which again contributes 1. This is depicted in Figure 9c. In both cases the other vertices contribute 3/2, as described for Figure 8d, thus summing up to at least 5/2. This covers the three newly introduced weak segments with ratio at least $\frac{5}{6}$. Figure 9d is a straightforward generalization of Figure 9b. The total new dual contribution is thus at least $\frac{7}{2}$, which covers the four new weak segments with ratio at least $\frac{7}{8}$. We do not depict the generalizations of Figure 9d, analogous to the one from Figure 9b to Figure 9c, which does not change the analysis. This completes the induction.

Figure 9b to Figure 9c, which does not change the analysis. This completes the induction. We now establish $\frac{|F_1|}{opt(G_1)} \leq \frac{4}{3}$. By definition, both the base case and the inductive step for a 2-ECSS are subsets of those for a 2-VCSS except the one depicted in Figure 8b. The set of configurations specific to a 2-VCSS are those depicted in Figure 7a, Figure 7b, and Figure 7c, where the weak segments are trivial segments. We have already analyzed these cases, and thus the proof is completed.

Theorem 8. $|F| \leq \frac{4}{3}opt(G)$. $|\overline{F}| \leq \frac{4}{3}opt(G)$.

Proof. Follows from Lemma 7 and the fourth claim of Lemma 3.



Figure 10: A tight example for the algorithm: (a) Input graph; (b) A solution returned by the algorithm; (c) An optimal solution.

5 A Tight Example

A tight example for the algorithm is given in Figure 10. The solution returned by the algorithm has cost 4k. The optimal solution has cost 3k + 2.

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